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Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. MME-5
Theory of Electromagnetic Research
Contract No. DA49-170-sc-2253
July, 1957

On the Measurement of Virtual Height

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Abstract

A time dependent definition of the virtual height of a reflected wave train is suggested. This definition is such that its accuracy increases as the width of the incident pulse increases. Moreover it is theoretically possible to obtain an estimate of the virtual height with as small an error as desired no matter what the nature of the reflecting medium.

Suppose an incident pulse having width W and a carrier frequency ω_o produces a reflected wave which, when measured at a fixed point x = 0 in space, is $R(\omega_o, W, t)$. The suggested definition of the virtual height $h'(\omega_o)$ is

$$h'(\omega_o) = \lim_{w \to \infty} c \left\{ \int_{-\infty}^{\infty} t|R(\omega_o,w,t)|^2\,dt \Big/ \int_{-\infty}^{\infty} |R(\omega_o,w,t)|^2 dt \right.$$

$$\left. - \int_{-\infty}^{\infty} t|I(\omega_o,w,t)|^2\,dt \Big/ \int_{-\infty}^{\infty} |I(\omega_o,w,t)|^2 dt \right\},$$

where c is the free space velocity of light. This relation for $h'(\omega_o)$ holds for any physically reasonable incident wave train and is independent of its envelope shape.

An example is given of a reflected wave whose virtual height cannot be determined by inspection in the usual manner. The expression given here for the virtual height provides the correct result.

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1. Introduction

Much of the data obtained from radio sounding investigations of the upper atmosphere is given in terms of the virtual height reached by a narrow pulse-modulated wave transmitted from the earth. A virtual height record, which is a record of the group delay time of the test pulse multiplied by the free space velocity of light, is generally made from measurements of the time required roughly for the wave packet to go out from and return to the antenna from which it was radiated.

Some ambiguity in the measurement of the delay time of the pulse will always occur in a standard experiment. The reason for this ambiguity lies in the difficulty of prescribing a fixed point of reference on the pulse envelope so that the movement of the wave group can be described as the movement of a single entity. The difficulty, of course, occurs because the pulse envelope must undergo a certain amount of distortion in its travel back and forth, including some spreading of its energy over a longer time interval than that which determined its inital width.

Still, in many cases the amount of distortion suffered by the pulse is small[1]. When such a case is under consideration the ambiguity in the virtual height measurement should be correspondingly small. It may be useful, however, to remark that in any case the ambiguity caused by the distortion of a test pulse in a radio sounding experiment is not inherent in this kind of measurement. It is due, rather, to the imprecise concept of group delay time used in the interpretation of the data supplied by the experiment. In the present paper an attempt will be made to overcome the difficulty by referring to the precise definition of group delay time instead of the heuristic idea of it.

The precise definition of virtual height is generally given as the free space velocity of light times the derivative with respect to frequency of the phase of the complex reflection coefficient which is characteristic of the medium being investigated[1],[2]. To be more explicit: imagine a c.w. of unit amplitude and frequency ω incident on the medium. The complex number $|r(\omega)| \exp\{j\phi(\omega)\}$, giving the amplitude $|r(\omega)|$ and the phase $\phi(\omega)$ of the reflected wave, is the reflection coefficient of the medium; the virtual height of a pulse at a frequency ω_0 for this medium will be

$$(1.1) \qquad h' = c \, d\phi/d\omega|_{\omega = \omega_0} \quad ,$$

where c is the free space velocity of light.

If this precise definition of virtual height is used, a second criticism of the standard measurement technique comes to mind. Due to the so-called uncertainty principle relating a transient wave form and its frequency spectrum, a very narrow pulse will have a spectrum which is rather evenly distributed over a broad frequency band; consequently, the idea of the group delay time of the pulse loses its intuitive meaning. The group delay time, then, will not necessarily be given correctly in terms of the phase derivative of the reflection coefficient, at least according to the arguments ordinarily found in literature[1], [2], [3]. In fact, in order to obtain a sensible definition of group delay time, one must deal with a wave packet having a narrow-band frequency spectrum, that is, a wave packet considerably spread out in time. Therefore, it seems reasonable that in order to improve the accuracy and to remove the ambiguity of virtual height measurements, one should use as broad a pulse as possible in radio sounding experiments.

It is the purpose of this paper to suggest the kind of measurement appropriate to the use of a broad test pulse instead of a narrow one. The analysis shows that theoretically, at least, the ambiguity in such a measurement can be made arbitrarily small for all carrier frequencies no matter what the dispersive properties of the reflecting medium may be. In particular, the $h' - f$ curves obtained at the present time from radio soundings of the ionosphere exhibit a great deal of ambiguity when the carrier frequency is near the penetration or critical frequency of a layer. Thus the use of a broad test pulse with the interpretation of virtual height suggested here offers the possibility for improving the accuracy of measurements at frequencies in the neighborhood of the critical frequency, for example.

2. Definitions

The proposed modification of the method for obtaining the virtual height is based on the following considerations.

Suppose, at a point $x = 0$ in free space below the medium to be investigated, the incident pulse is given by a real function of time t

$$I(\omega_o, w, t) = I(t/w) \cos \omega_o t, \tag{2.1}$$

where I (t/w) represents a more or less arbitrary pulse envelope with a maximum at $t = 0$. The constant w, assumed positive, roughly fixes the width of the pulse in the sense that as w becomeslarge I(t/w) remains near its value at $t = 0$ for a large range of t. For example, we might have

$$I(t/w) = \exp (- t^2/2w^2) , \tag{2.2}$$

a Gaussian envelope with standard deviation w or

$$I(t/w) = \begin{cases} 1 & \text{for } -w/2 < t < w/2 \\ 0 & \text{for } |t| > w/2 \end{cases} , \tag{2.3}$$

a square pulse of width w.

If we define the Fourier transform

$$\begin{aligned} i(\omega) &= \int_{-\infty}^{\infty} I(\tau) \exp(j\omega\tau)d\tau, \quad i(\omega) = |i(\omega)| \exp\{j\theta(\omega)\} \\ I(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} i(\omega) \exp(-j\omega\tau) \, d\omega , \end{aligned} \tag{2.4}$$

then the Fourier transform of the incident pulse (2.1) is the function

$$\begin{aligned} i(\omega_o, w, \omega) &= \int_{-\infty}^{\infty} I(\omega_o, w, t) \exp (j\omega t)dt \\ &= (w/2)\left[i\{w(\omega + \omega_o)\} + i\{w(\omega - \omega_o)\}\right] . \end{aligned} \tag{2.5}$$

In terms of the reflection coefficient $r(\omega)$ the Fourier transform of the reflected wave packet $R(\omega_o, w,t)$ is $r(\omega)\ \ i(\omega_o, w,\omega)$ and we have

$$(2.6)\qquad R(\omega_o, w,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r(\omega)\ \ i(\omega_o, w,\omega) \exp(-jwt)d\omega.$$

It will be shown later that under these conditions:

$$\lim_{w\to\infty}\int_{-\infty}^{\infty}|R(\omega_o, w,t)|^2\,dt\Big/\int_{-\infty}^{\infty} I(\omega_o, w,t)|^2\,dt = |r(\omega_o)|^2$$

and

$$(2.7)\qquad \lim_{w\to\infty} c\left\{\int_{-\infty}^{\infty} t|R(\omega_o, w,t)|^2\,dt\Big/\int_{-\infty}^{\infty}|R(\omega_o,w,t)|^2\,dt - \int_{-\infty}^{\infty} t|I(\omega_o,w,t)|^2\,dt\Big/\int_{-\infty}^{\infty}|I(\omega_o,w,t)|^2\,dt\right\}$$

$$= c\, d\phi(\omega)/d\omega\big|_{\omega=\omega_o} = h'(\omega_o) \qquad \text{(by 1.1)).}$$

The relations (2.7) indicate that except for quantities of infinitesimal order as w becomes large the amplitude of the reflection coefficient and the virtual height of the incident pulse can be expressed in terms of the power and the centroid of the power in the incident and reflected wave packets. The error committed in using (2.7) with large but finite w would seem to depend on $\omega_o w$ as well as on the ratio of w to the effective width of the reflecting medium.

In the case of a narrow incident pulse which produces a narrow reflected pulse the measurement of virtual height obtained by the standard technique (consisting of a direct physical observation of the delay time) will give the same result as the approximation based on (2.7) where the actual finite value of w is used instead of $w\to\infty$. Thus, the approximation suggested by (2.7) will always be at least as accurate as the standard method for measuring virtual height and should be more accurate when a broad incident pulse is used instead of a narrow one.

3. Example of a dispersed reflected pulse

It is easy to find reflection coefficients in connection with a given incident pulse of any desired width such that an inspection of the corresponding reflected wave packet will reveal no clue to the virtual height as defined by (1.1). If the incident pulse is sufficiently broad, however, the computation suggested by (2.7) will always give the virtual height even in this case.

Consider for example, a reflection coefficient

$$r(\omega) = j\alpha \exp(2j\omega L)/(\omega + j\alpha) \quad , \tag{3.1}$$

where α and L are real positive constants. The amplitude of $r(\omega)$ is

$$|r(\omega)| = \alpha/(\omega^2 + \alpha^2)^{1/2} \quad ; \tag{3.2}$$

the phase of $r(\omega)$ is

$$\phi(\omega) = -\tan^{-1}(\alpha/\omega) + \pi/2 + 2\omega L \quad , \tag{3.3}$$

and thus the virtual height, according to (1.1), is

$$h'(\omega_o) = c\alpha/(\omega_o^2 + \alpha^2) + 2cL \quad . \tag{3.4}$$

Let us choose for the incident pulse

$$I(\omega_o, w, \omega) = I(t/w) \cos \omega_o t \quad , \tag{3.5}$$

where

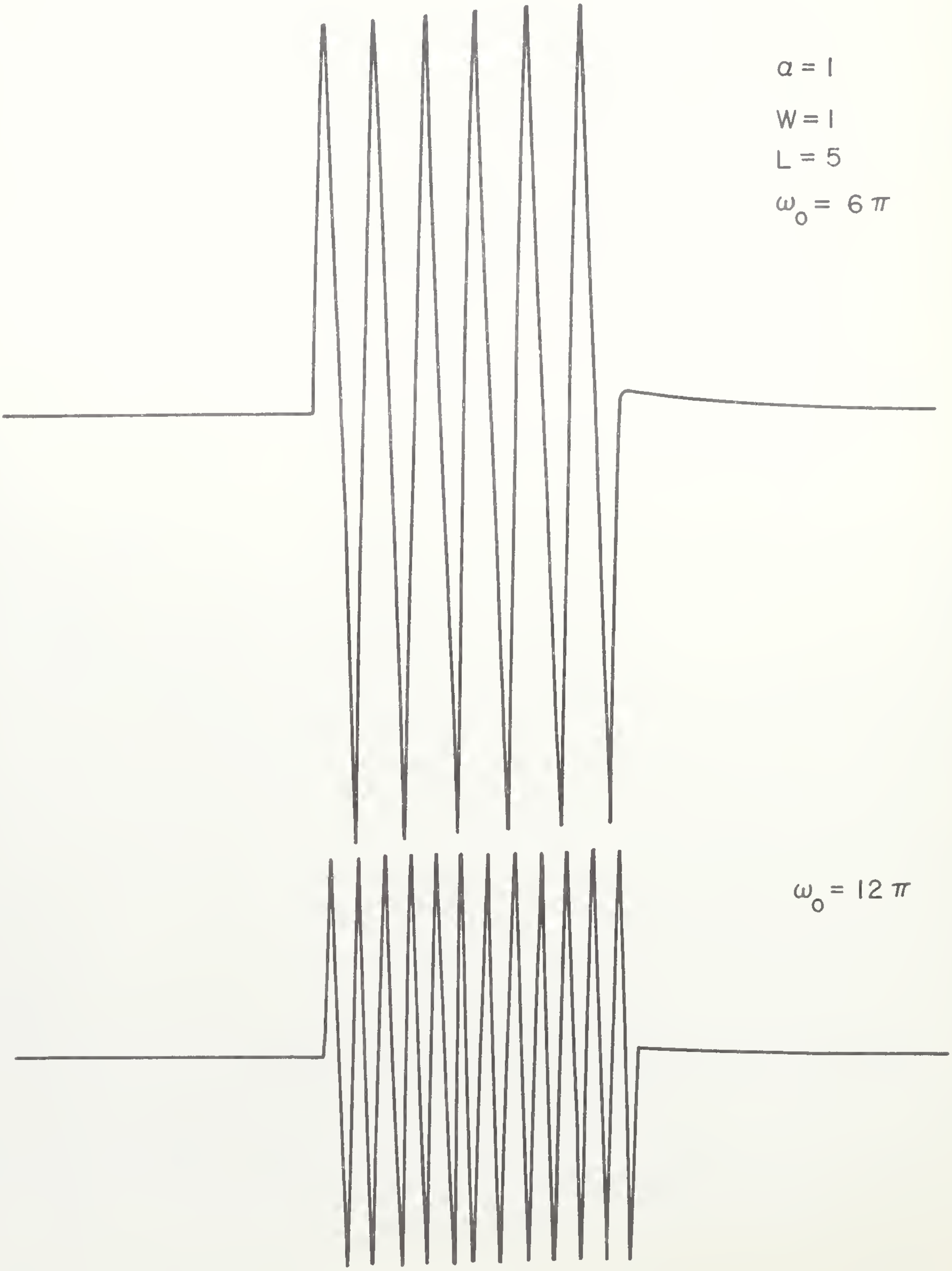
$$I(t/w) = \begin{cases} 1, & \text{for } |t| < w \\ 0, & \text{for } |t| > w \end{cases} \quad . \tag{3.6}$$

The reflected wave packet is given by the expression

$$(3.7)\quad R(\omega_o, w, t) = \begin{cases} 0, & t<2L-w \\ \sin\theta\left[\exp\{a(2L-w-t)\}\sin(\omega_o w-\theta)+\sin\{\omega_o(t-2L)+\theta\}\right], & 2L-w<t<2L+w \\ \sin\theta\left[\exp\{a(2L+w-t)\}\sin(\omega_o w+\theta)+\exp\{a(2L-w-t)\}\sin(\omega_o w-\theta)\right], & 2L+w<t\ , \end{cases}$$

where $\sin\theta = a/(a^2+\omega_o^2)^{1/2}$.

REFLECTED PULSE $R(\omega_o, W, t)$ VS TIME t



The envelope of $R(\omega_o, w, t)$ has essentially the same shape for any value of the carrier frequency ω_o. Thus, there is no obvious way to obtain the virtual height (3.4) from $R(\omega_o, w, t)$ by inspection.

We can apply (2.7) to compute the virtual height, however. We have:

(3.8) $$i(\omega) = \int_{-1}^{1} \exp\,(j\omega t)dt = 2\,\sin\omega\,/\omega$$

so that by (2.5)

(3.9) $$i(\omega_o, w, \omega) = \sin\left\{w(\omega+\omega_o)\right\}/(\omega+\omega_o) + \sin\left\{w(\omega-\omega_o)\right\}/(\omega-\omega_o) \quad .$$

We also have:

$$\int_{-\infty}^{\infty} |I(\omega_o, w, t)|^2\, dt = w + \sigma(w)$$

(3.10) $$\int_{-\infty}^{\infty} |R(\omega_o, w, t)|^2\, dt = w\alpha^2/(\alpha^2 + \omega_o^2) + \sigma(w)$$

$$\int_{-\infty}^{\infty} t\,|R(\omega_o, w, t)|^2\, dt = w\alpha^3/(\omega_o^2 + \alpha^2)^2 + 2wL\alpha^2/(\alpha^2 + \omega_o^2) + \sigma(w) \quad .$$

The general results (2.7) are thus verified in this special case. The error actually depends on $\omega_o w$ and αw but not particularly on L, as a more detailed calculation will show. The quantity L, according to (3.7), is just the distance of the edge of the reflecting medium divided by c, the free space velocity of light. The quantity α may be regarded as a measure of the width of the reflecting medium since $r(\omega)$ becomes zero if α approaches zero.

4. Proof of (2.7):

Relations and definitions (2.4), (2.5) and (2.6) are necessary for the proof of (2.7). A standard argument which is employed in the proof of the Fourier integral theorem will be used to demonstrate a preliminary result we shall require.

Theorem A: Let $f(\omega)$ *be continuous and uniformly bounded for all real* ω, *and assume*

$$\int_{-\infty}^{\infty} |I(\tau)|^2 \, d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} |i(\omega)|^2 \, d\omega < \infty \quad .$$

Then

$$\int_{-\infty}^{\infty} f(\omega) |i(\omega_o, w, \omega)|^2 \, d\omega = \frac{w}{4} \left\{ f(\omega_o) + f(-\omega_o) \right\} \int_{-\infty}^{\infty} |i(\omega)|^2 d\omega + \sigma(w) \quad .$$

Proof: By (2.5) we have

$$|i(\omega_o, w, \omega)|^2 = \frac{w^2}{4} \Big[|i\{w(\omega+\omega_o)\}|^2 + |i\{w(\omega-\omega_o)\}|^2 + i\{w(\omega+\omega_o)\} i^*\{w(\omega-\omega_o)\}$$

(4.1)

$$+ i^*\{w(\omega+\omega_o)\} i\{w(\omega-\omega_o)\} \Big] \quad .$$

First we consider the first two terms on the right of (4.1) at the same time. We have

$$\int_{-\infty}^{\infty} \left\{ f(\omega) - f(\pm\omega_o) \right\} |i\{w(\omega \pm \omega_o)\}|^2 d\omega = \frac{1}{w} \int_{-\infty}^{\infty} \left\{ f(\sigma/w \pm \omega_o) - f(\pm\omega_o) \right\} |i(\sigma)|^2 d\sigma$$

(4.2)

$$= \frac{1}{w} \int_{-\sqrt{w}}^{\sqrt{w}} \left\{ f(\sigma/w \pm \omega_o) - f(\pm\omega_o) \right\} |i(\sigma)|^2 d\sigma + \frac{1}{w} \left[\int_{\sqrt{w}}^{\infty} + \int_{-\infty}^{-\sqrt{w}} \right] \left\{ f(\sigma/w \pm \omega_o) - f(\pm\omega_o) \right\} |i(\sigma)|^2 d\sigma .$$

Consider the first term on the right of (4.2). We have

$$(4.3) \qquad \frac{1}{w} \int_{-\sqrt{w}}^{\sqrt{w}} |f(\sigma/w \pm \omega_o) - f(\pm\omega_o)| \, |i(\sigma)|^2 d\sigma \leq \frac{\varepsilon}{w} \int_{-\infty}^{\infty} |i(\sigma)|^2 d\sigma$$

for w large enough, by the continuity of $f(\omega)$ at ω_o. The last two terms on the right of (4.2) are evidently equal to $(1/w)\ \sigma(1)$. Since ε in (4.3) can be made as small as we please for large w, we have finally

$$(4.4)\qquad \int_{-\infty}^{\infty} f(\omega)\left|i\left\{w(\omega\pm\omega_0)\right\}\right|^2 d\omega = (1/w)f(\pm\omega_0)\int_{-\infty}^{\infty} |i(\sigma)|^2 d\sigma + (1/w)\sigma(1) \quad .$$

Now we consider the last two terms on the right of (4.1) one at a time. We have

$$\int_{-\infty}^{\infty} f(\omega)i\left\{w(\omega+\omega_0)\right\}i^*\left\{w(\omega-\omega_0)\right\}d\omega$$

$$(4.5)\qquad = (1/w)\int_{-\infty}^{\infty} f(\sigma/w+\omega_0)i^*(\sigma)i(\sigma+2\omega_0 w)d\sigma$$

$$= (1/w)\int_{-\sqrt{w}}^{\substack{-\infty\\ \sqrt{w}}} f(\sigma/w+\omega_0)i^*(\sigma)i(\sigma+2\omega_0 w)d\sigma$$

$$+ (1/w)\left[\int_{\sqrt{w}}^{\infty} + \int_{-\infty}^{-\sqrt{w}}\right] f(\sigma/w+\omega_0)i^*(\sigma)i(\sigma+2\omega_0 w)d\sigma \quad .$$

By Schwartz's inequality the first term of the right side of (4.5) is of order

$$(1/w)\left[\int_{-\infty}^{\infty} |i(\sigma)|^2 d\sigma\right]^{1/2}\left[\int_{2\omega_0 w-\sqrt{w}}^{2\omega_0 w+\sqrt{w}} |i(\sigma)|^2 d\sigma\right]^{1/2} = (1/w)\sigma(1) \quad .$$

The second and third terms on the right side of (4.5) can also be estimated by Schwartz's inequality:

$$(1/w)\int_{\sqrt{w}}^{\infty} f(\sigma/w+\omega_0)i^*(\sigma)i(\sigma+2\omega_0 w)d\sigma$$

$$\leq (1/w)\left[\int_{\sqrt{w}}^{\infty} |i(\sigma)|^2 d\sigma \int_{\sqrt{w}}^{\infty} |f(\sigma/w+\omega_0)|^2 |i(\sigma+2\omega_0 w)|^2 d\sigma\right]^{1/2} = (1/w)\sigma(1) \;,$$

and of course we have the same result for the other term. Combining these results with (4.1) we arrive at Theorem A.

From Plancherel's theorem for the Fourier transform (2.6) and the fact that $|r(\omega)|$ is an even function of ω, we obtain the first part of (2.7). To obtain the second part we start with Plancherel's theorem applied to the first moment of a function. Thus, if $f(\omega)$ is the Fourier transform of $F(t)$ we note that

$$\int_{-\infty}^{\infty} t|F(t)|^2 \, dt = \frac{-j}{2\pi} \int_{-\infty}^{\infty} f'(\omega) f^*(\omega) d\omega \quad . \tag{4.6}$$

Applying (4.6) to the numerator of the first term on the left of the second equation in (2.7) we have

$$\begin{aligned}
\int_{-\infty}^{\infty} t|R(\omega_o,W,t)|^2 \, dt &= \frac{-j}{2\pi} \int_{-\infty}^{\infty} \frac{d}{d\omega}\left\{r(\omega) i(\omega_o,W,\omega\right\} \cdot r^*(\omega) i^*(\omega_o,W,\omega) d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{d\omega}\left\{\emptyset(\omega)+\theta(\omega_o,W,\omega)\right\} |i(\omega_o,W,\omega)|^2 \, |r(\omega)|^2 \, d\omega \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{d\omega}\left\{|r(\omega)|\,|i(\omega_o,W,\omega)|\right\} \left\{|r(\omega)|\,|i(\omega_o,W,\omega)|\right\} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d}{d\omega} \left\{\emptyset(\omega)+\theta(\omega_o,W,\omega)\right\} |i(\omega_o,W,\omega)|^2 \, |r(\omega)|^2 \, d\omega \quad ,
\end{aligned} \tag{4.7}$$

for

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d}{d\omega} \left\{|r(\omega)|\,|i(\omega_o,W,\omega)|\right\} \left\{|r(\omega)|\,|i(\omega_o,W,\omega)|\right\} d\omega \\
= \frac{1}{2} \int_{-\infty}^{\infty} \frac{d}{d\omega} \left\{|r(\omega)|^2 \, |i(\omega_o,W,\omega)|^2\right\} d\omega = 0.
\end{aligned}$$

A similar consideration applies to the numerator of the second term on the left of the second equation in (2.7). Combining these results with an application of Theorem A, we obtain the second relation in (2.7).

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